Irrotational motion associated with free turbulent flows

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Summary

The irrotational motions induced by, but outside of, the self-preserving turbulent wake and jet are examined. It is found that there is a mean flow towards the centre of the jet, although there is no such flow in the case of the wake. Phillips's (1955) results on the nature of fluctuating irrotational flows are found to be largely unaffected by the introduction of simple inhomogeneities into the boundary conditions, but another mode of fluctuation with movement along the lines of mean flow is also shown to be likely. It is pointed out that, contrary to a statement of Corrsin & Kistler (1954), it is possible for the vorticity-free fluid between bulges of turbulent fluid to partake of the mean velocity of the turbulent fluid.

1. Introduction

Experimental researches in recent years have established beyond doubt that in all those flows, such as wakes, jets and boundary layers, where turbulence is expanding into non-turbulent regions of fluid, there exists a relatively sharp, although violently contorted, boundary separating the fully turbulent region and an external region. This external region, although not stationary, is characterized by very much smaller velocity gradients than is the turbulent region. Corrsin & Kistler (1954) and Townsend (1956) have pointed out that since vorticity can be introduced into non-vortical fluid only by viscosity, which is a very short range force, the motion in this external region is probably irrotational, and therefore can be derived from a scalar potential.

Phillips (1955) has considered a simple case of such irrotational motion, taking boundary conditions such that the velocity component u_2 is a stationary random function of the cartesian coordinates x_1, and x_3 on the plane $x_2 = 0$* and that the velocity $\mathbf{u}$ vanishes as $x_2 \to \infty$. It is the purpose of this paper to extend Phillips's results to some situations in which the statistical properties of u_2 are not independent of both x_1 and x_3, and to offer some comments and speculations on the phenomena.

* We have changed the order of the indices from that used by Phillips in order to conform to customary usage in wakes and jets. x_1 is the principal direction of flow and x_3 is parallel to any direction of symmetry.

We begin with a review of Phillips's analysis. Given that

$$(u_2)_{x_2=0} = \int e^{i\mathbf{k}.\mathbf{x}}\, dA(\mathbf{k}),$$

where $\mathbf{k}$ is a (two-dimensional) vector wave-number with components $(k_1, 0, k_3)$ and magnitude k, $\mathbf{x}$ is the position vector with components (x_1, x_2, x_3) and the integration is over all $\mathbf{k}$-space (two-dimensional); and defining

$$\theta(\mathbf{k}) = \lim_{d\mathbf{k}\to 0} \frac{\overline{dA(\mathbf{k})\, dA^*(\mathbf{k})}}{dk_1\, dk_3},$$

where the bar indicates an average over the plane $x_2 = 0$ and the asterisk denotes the complex conjugate; he finds that, for all $x_2 > 0$,

$$\left.\begin{aligned} \overline{u_2^2} &= \int \theta(\mathbf{k}) e^{-2kx_2}\, d\mathbf{k}, \\ \overline{u_i^2} &= \int \frac{k_i^2}{k^2} \theta(\mathbf{k}) e^{-2kx_2}\, d\mathbf{k} \qquad (i = 1, 3). \end{aligned}\right\} \tag{1.1}$$

It is shown that, since $\theta(\mathbf{k})$ is an even function,

$$\overline{u_1 u_2} = \int \frac{k_1}{k} \theta(\mathbf{k}) e^{-2kx_2}\, d\mathbf{k} = 0 = \overline{u_2 u_3}. \tag{1.2}$$

He also states that $\overline{u_1 u_3} = 0$, and indicates that a similar proof would establish this relation. This is not the most general case, however, for in fact

$$\overline{u_1 u_3} = \int \frac{k_1 k_3}{k^2} \theta(\mathbf{k}) e^{-2kx_2}\, d\mathbf{k}, \tag{1.3}$$

which is not generally zero. Since $\overline{u_i u_j}$ is a symmetric tensor there will always, of course, be a possible choice of the x_1 and x_3-coordinate directions for which $\overline{u_1 u_3} = 0$. In the ordinary cases of the two-dimensional wake, jet or boundary layer, this choice corresponds to the 'natural' one of x_1 as the downstream direction. For the wake of a yawed cylinder or the boundary layer over a plate whose leading edge is not perpendicular to the flow, however, $\overline{u_1 u_3} \neq 0$ if the x_1-direction is chosen to be the principal direction of flow. There is no need to exclude such examples, and we may continue the analysis in the manner of Phillips.

Assuming that $\theta(\mathbf{k})$ may be expanded near $\mathbf{k} = 0$ in the form

$$\theta(\mathbf{k}) = \theta + k_i \theta_i + k_i k_j \theta_{ij} + k_i k_j k_l \theta_{ijl} + \ldots,$$

where θ, θ_i, θ_{ij}, etc. are cartesian tensors of the indicated order and are independent of the k_i, Phillips shows that $\theta = \theta_i = 0$. At large values of x_2 the values of $\theta(\mathbf{k})$ contributing significantly to the integrals in equations (1.1) to (1.3) are given approximately by

$$\theta(\mathbf{k}) \sim k_i k_j \theta_{ij}. \tag{1.4}$$

Putting (1.4) in (1.3) we get

$$\begin{aligned} \overline{u_1 u_3} &\sim \int \frac{k_1 k_3}{k^2} k_i k_j \theta_{ij} e^{-2kx_2}\, d\mathbf{k} \\ &= \int \frac{k_1^2 k_3^2}{k^2} (\theta_{13} + \theta_{31}) e^{-2kx_2}\, d\mathbf{k} \end{aligned} \tag{1.5}$$

since terms in θ_{11} and θ_{33} evidently vanish for reasons of symmetry. There is no loss of generality if we take θ_{ij} to be symmetric, so

$$\overline{u_1 u_3} \sim 2 \int \frac{k_1^2 k_3^2}{k^2} \theta_{13} e^{-2kx_2} d\mathbf{k} = \tfrac{3}{16}\pi\theta_{13} x_2^{-4}. \tag{1.6}$$

We also write down for reference relations derived by Phillips:

$$\left.\begin{aligned} \overline{u_2^2} &\sim \tfrac{3}{8}\pi(\theta_{11}+\theta_{33})x_2^{-4}, \\ \overline{u_1^2} &\sim \tfrac{3}{32}\pi(3\theta_{11}+\theta_{33})x_2^{-4}, \\ \overline{u_3^2} &\sim \tfrac{3}{32}\pi(\theta_{11}+3\theta_{33})x_2^{-4}. \end{aligned}\right\} \tag{1.7}$$

Thus we see that in general there *is* a Reynolds shear stress due to the irrotational motion, with its direction parallel to the plane $x_2 = 0$.

It is evident from (1.1) that

$$\overline{u_2^2} = \overline{u_1^2} + \overline{u_3^2}, \tag{1.8}$$

so that the velocity fluctuations are not isotropic. This is of some consequence, for in the following sections we shall consider situations where u_2 is not statistically stationary on the plane $x_2 = 0$, and it can be shown that this lack of istropy then gives rise to stress gradients. Thus, in the absence of viscosity the total cartesian stress tensor in an incompressible fluid is

$$\tau_{ij} = P\delta_{ij} + \overline{u_i u_j} \tag{1.9}$$

where P is the pressure, δ_{ij} is the Kronecker delta, and $\overline{u_i u_j}$ is the Reynolds stress. If the fluctuating velocities are distributed isotropically, this becomes

$$\tau_{ij} = (P + \overline{u^2})\delta_{ij}, \tag{1.10}$$

and it is possible for gradients in $\overline{u^2}$ to be compensated by gradients in P, so that no net stress gradients (or accelerations) need result. In the absence of istropy this compensation becomes impossible and variations in turbulent intensity are necessarily accompanied by stress gradients.

In addition, Corrsin & Kistler (1954) have shown that in the special case of irrotational flow the relation

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \tag{1.11}$$

imposes conditions on the Reynolds stresses which have some interesting implications. For example, expression (1.8) can be derived very simply from the boundary conditions and the relation (1.11). Thus the Reynolds stress gradients must satisfy the relations

$$\frac{\partial}{\partial x_j}\overline{u_2 u_j} = \overline{u_j \frac{\partial u_2}{\partial x_j}} = \overline{u_j \frac{\partial u_j}{\partial x_2}} = \frac{1}{2}\frac{\partial}{\partial x_2}\overline{u_j u_j}.$$

But the fluctuations are stationary in the x_1- and x_3-directions. Hence

$$\frac{\partial \overline{u_1 u_2}}{\partial x_1} = \frac{\partial \overline{u_2 u_3}}{\partial x_3} = 0,$$

and so
$$\frac{\partial \overline{u_2^2}}{\partial x_2} = \frac{1}{2}\frac{\partial}{\partial x_2}\overline{u_j u_j}.$$

Then, since $\mathbf{u}$ vanishes as $x_2 \to \infty$,
$$\overline{u_2^2} = \tfrac{1}{2}\overline{u_j u_j} = \overline{u_1^2}+\overline{u_3^2}.$$

2. Cylinder wake

The predictions of Phillips's theory have been remarkably confirmed by comparison with measurements, made by A. A. Townsend in the wake of a circular cylinder. It is probably worth examining the potential flow associated with such a wake in more detail.

We shall consider only a self-preserving flow which Townsend (1956) has shown to be possible only if the turbulent velocities are small compared with the mean flow velocity U_∞. In this case the characteristic scale L varies as $x_1^{1/2}$ when the x_1-direction is the principal direction of flow and the origin of x_1 is some virtual origin not too far from the actual location of the cylinder. It is also shown that, for the transport of momentum to be independent of x_1, $U_0 \propto x_1^{-1/2}$, where U_0 is some characteristic velocity which may be taken as $U_0 = U_\infty - (U_1)_{\min}$ for any given x_1. Root-mean-square turbulent velocities vary as U_0.

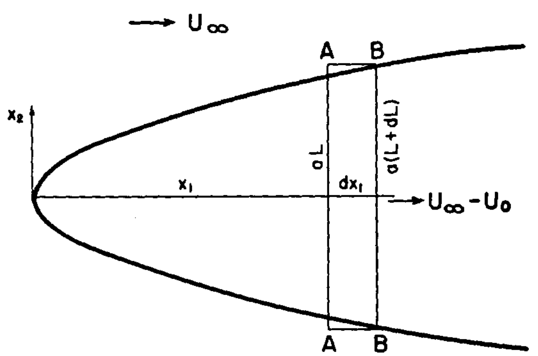


Figure 1. The self-preserving wake.
The scale in the x_2-direction is greatly expanded.

In figure 1 the total mean transport of fluid across the section AA per unit length in the x_3-direction is
$$aL(U_\infty - KU_0) + a\,dL\,U_\infty,$$
where KU_0 is the average value of $U_\infty - U_1$ across the wake, the total extent of which is aL, and K and a are constant. The total mean transport of fluid across the section BB is
$$a(L+dL)\{U_\infty - K(U_0+dU_0)\}.$$

Neglecting second order differentials, the transport across AB in the x_2-direction is therefore

$$\tfrac{1}{2}aK(L\,dU_0 + U_0\,dL) = \tfrac{1}{2}aK\,d(U_0\,L).$$

Hence

$$U_2 \propto \frac{d}{dx_1}(U_0\,L) = 0. \tag{2.1}$$

Thus we find that there is no mean flow from the x_2-direction into a self-preserving two dimensional wake. We shall see later that this is not true of a jet.

3. Fluctuating motions outside a turbulent wake

Returning to the case considered by Phillips, let us assume that it is possible to define $\theta(\mathbf{k}, x_1)$ for every value of x_1 on some plane* $x_2 = 0$ outside, but close to, the main body of the wake. We assume that, although $\theta(\mathbf{k}, x_1)$ is inhomogeneous in x_1, Phillips' main conclusions are not materially altered.

Our similarity conditions require that

$$\theta(\mathbf{k}, x_1) = \chi(\mathbf{k}x_1{}^{1/2}) = \chi(\boldsymbol{\epsilon}), \tag{3.1}$$

where χ is a function of $\boldsymbol{\epsilon} = \mathbf{k}x_1^{1/2}$ only, in order that the characteristic scales should vary as $x_1^{1/2}$, and that the energy per unit mass in the fluctuations,

$$\int \theta(\mathbf{k}, x_1)\,d\mathbf{k} = x_1^{-1}\int \chi(\boldsymbol{\epsilon})\,d\boldsymbol{\epsilon},$$

should vary as x_1^{-1}. The expansion

$$\theta(\mathbf{k}, x_1) = k_i k_j \theta_{ij}(x_1) + k_i k_j k_l k_m \theta_{ijlm}(x_1) + \ldots$$

may then be written

$$\theta(\mathbf{k}, x_1) = x_1 k_i k_j \chi_{ij} + x_1^2 k_i k_j k_l k_m \chi_{ijlm} + \ldots, \tag{3.2}$$

where χ_{ij}, etc. are absolute constants for the particular flow.

At sufficiently large values of x_2, we find from (1.7), (1.8) and (3.2) that

$$\left.\begin{aligned} \overline{u_2^2} &\sim \tfrac{3}{8}\pi(\chi_{11}+\chi_{33})x_1 x_2^{-4}, \\ \overline{u_1^2} &\sim \tfrac{3}{32}\pi(3\chi_{11}+\chi_{33})x_1 x_2^{-4}, \\ \overline{u_3^2} &\sim \tfrac{3}{32}\pi(\chi_{11}+3\chi_{33})x_1 x_2^{-4}. \end{aligned}\right\} \tag{3.3}$$

By symmetry

$$\overline{u_1 u_3} = \overline{u_2 u_3} = 0.$$

It is interesting to note that these fluctuation intensities *increase* as x_1 increases. The reason for this, of course, is that the increase in the scale of the primary causative turbulence more than compensates for the decrease in intensity. If we consider the variation on similarity surfaces $x_2 \propto x_1^{1/2}$, instead of on planes $x_2 =$ constant, we find that the fluctuation intensities are all proportional to x_1^{-1}, as they should be.

The equations for incompressible mean flow in the steady state are

$$U_k\frac{\partial U_i}{\partial x_k} + \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_k}\overline{u_i u_k} - \nu\frac{\partial^2 U_i}{\partial x_k \partial x_k} = 0. \tag{3.4}$$

* The actual surface should be parabolic, but this introduces considerable complication into the analysis and at large values of x_1 the difference should be slight.

If we have Reynolds number similarity (Townsend 1956) which we shall assume, the viscous term will be negligible. In the particular region of the flow in which we are interested we have seen that

$$U_2 \sim U_3 = \overline{u_1 u_3} = \overline{u_2 u_3} = 0,$$

and we shall see later (3.9) that $(\partial/\partial x_1)\overline{u_1 u_2} = 0$; so it would appear reasonable to write for the x_2-component

$$\frac{1}{\rho}\frac{\partial P}{\partial x_2} + \frac{\partial \overline{u_2^2}}{\partial x_2} = 0,$$

or

$$P + \rho\overline{u_2^2} = P_\infty, \tag{3.5}$$

P_∞ being the value of P at large values of x_2, where $\overline{u_2^2}$ is very small.

For the x_1-component, the ony terms of any consequence will be

$$U_2\frac{\partial U_1}{\partial x_2} + U_1\frac{\partial U_1}{\partial x_1} + \frac{1}{\rho}\frac{\partial P}{\partial x_1} + \frac{\partial}{\partial x_2}\overline{u_1 u_2} + \frac{\partial \overline{u_1^2}}{\partial x_1} = 0. \tag{3.6}$$

Corrsin & Kistler (1954) have shown that, for irrotational flow

$$\frac{\partial \overline{u_i u_j}}{\partial x_j} = \frac{1}{2}\frac{\partial q^2}{\partial x_i}, \tag{3.7}$$

where

$$q^2 = \overline{u_j u_j}.$$

If mean quantities are independent of x_3, (3.7) gives

$$\left.\begin{aligned} \frac{\partial \overline{u_1 u_2}}{\partial x_1} &= \frac{1}{2}\frac{\partial}{\partial x_2}(q^2 - 2\overline{u_2^2}), \\ \frac{\partial \overline{u_1 u_2}}{\partial x_2} &= \frac{1}{2}\frac{\partial}{\partial x_1}(q^2 - 2\overline{u_1^2}). \end{aligned}\right\} \tag{3.8}$$

In our case, where $\overline{u_2^2} = \overline{u_1^2} + \overline{u_3^2} = \frac{1}{2}q^2$, this yields

$$\frac{\partial \overline{u_1 u_2}}{\partial x_1} = 0, \tag{3.9}$$

$$\frac{\partial \overline{u_1 u_2}}{\partial x_2} = \frac{\partial}{\partial x_1}(\overline{u_2^2} - \overline{u_1^2}), \tag{3.10}$$

and introducing (3.5) into (3.10) we obtain

$$\frac{\partial \overline{u_1 u_2}}{\partial x_2} = -\frac{\partial}{\partial x_1}\left(\frac{P}{\rho} + \overline{u_1^2}\right). \tag{3.11}$$

Combining (3.6) and (3.11) we find that

$$U_2\frac{\partial U_1}{\partial x_2} + U_1\frac{\partial U_1}{\partial x_1} = 0. \tag{3.12}$$

We therefore see that the mean flow is unaffected by the irrotational fluctuations, not because the Reynolds stresses are zero, but because the gradient of the shear stress $(\partial/\partial x_2)\overline{u_1 u_2}$ is exactly balanced by the gradient of the total normal stress $(\partial/\partial x_1)(P/\rho + \overline{u_1^2})$. It would appear that such a balance is the rule rather than the exception in irrotational fluctuating flows.

The normal stress gradient tends to accelerate fluid in the direction of the mean flow. However, the shear stress transfers this acceleration to the retarded flow in the main body of the wake. The whole process must be considered as a part of the mechanism by which this retarded fluid is accelerated towards U_∞ as the wake decays.

4. Motion between bulges of turbulent fluid

There is a difference of opinion expressed in the literature as to the nature of the mean flow in that portion of the external, irrotational flow region which lies between out-thrusts or bulges of the turbulent fluid. Townsend (1949) states that "the non-turbulent fluid between successive jets is constrained by pressure gradients to move at the same mean velocity as the fluid in the adjacent jets", and shows measurements (Townsend 1956, p. 163) to support this view. Corrsin & Kistler (1954), arguing that the mean flow in these regions is irrotational and therefore gradient free, say "the mean velocity everywhere in the potential field must be constant and equal to that at infinity" and publish measurements qualitatively substantiating this statement.

An examination of Phillips's results may cast some light on this controversy. It is noteworthy that equations (1.1) exactly correspond to the classical theory of flow in gravity water waves (Lamb 1932, ch. 9) if we consider a situation where there is a continuous spectrum of wave frequencies and waves may be oriented in all possible directions. Now it is well known (Lamb 1932, p. 418) that, although such waves are purely irrotational flows, if they have finite amplitude they are accompanied by a mean transport of fluid, and a resulting mean shear. The reason for this apparent contradiction is that they are bounded by a non-planar surface, and any plane passing between the level of maximum elevation and the level of maximum depression includes regions in which the equations of irrotational flow do not apply. The simple case of a single wave train with a single frequency is sketched in figure 2(*a*), which clearly illustrates the phenomenon. Everywhere below the boundary the flow is irrotational. The velocity potential is

$$\phi = -B\cos kx_1\, e^{-kx_2}, \tag{4.1}$$

where B is a constant, and the velocity components are

$$u_1 = Bk\sin kx_1\, e^{-kx_2},$$
$$u_2 = Bk\cos kx_1\, e^{-kx_2}.$$

For any $x_2 > 0$,

$$\overline{u_1} = \frac{Bk}{2\pi}\int_{-\pi/k}^{\pi/k} \sin kx_1\, e^{-kx_2}\, dx_1 = 0, \tag{4.2}$$

but, if the boundary* is described by

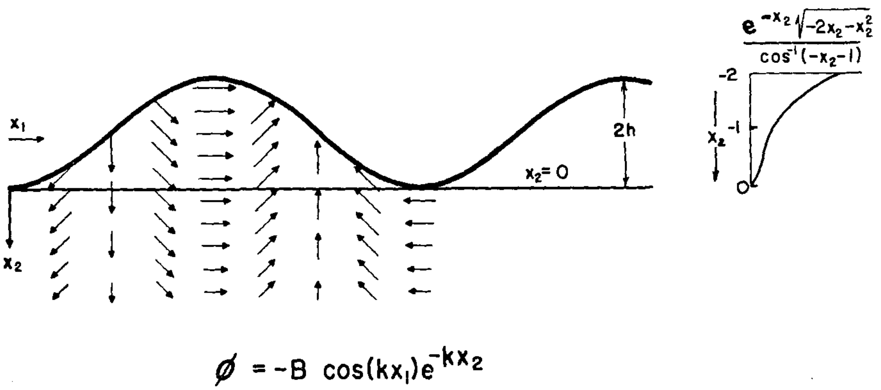
$$x_2 = -h(1+\sin kx_1), \tag{4.3}$$

* The fact that the actual boundary of a theoretical water wave is trochoidal does not concern us here.

for $0 > x_2 > -2h$, we find that,

$$\overline{u_1} = \frac{Bke^{-kx_2}}{\cos^{-1}(-1-x_2/h)} \int_{\pi/2k}^{\{\sin^{-1}(-1-x_2/h)\}/k} \sin kx_1\, dx_1$$
$$= \frac{Be^{-kx_2}}{\cos^{-1}(-1-x_2/h)} \sqrt{\left(\frac{2x_2}{h} - \frac{x_2^2}{h^2}\right)}. \tag{4.4}$$

This function is plotted in figure 2(*b*) for $B = k = h = 1$.



(*a*) (*b*)

Figure 2. (*a*) Flow in a simple gravity wave advancing from left to right. The length of the arrows is proportional to the logarithm of the velocity at their centres ; (*b*) the mean horizontal velocity within the wave.

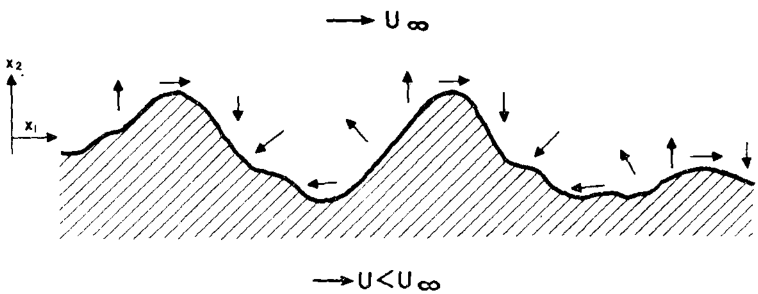


Figure 3. Probable directions of potential flow near the boundary of turbulent fluid in a wake.

The irrotational flow induced by free turbulence is another example of potential flow with an irregular boundary beyond which Laplace's equation does not hold. It is quite probable that a similar phenomenon occurs, for the pressure forces are likely to make the mean direction of the potential flow between bulges the same as that of the bulges themselves. The example of gravity waves shows that such a motion is *not* inconsistent with the equations of irrotational flow. If the flow is of this type, we can make some predictions concerning the direction of u_2 in the non-turbulent region (figure 3). In a wake this component should consistently be negative just before the arrival of a bulge and positive just after its passage. Perhaps the similarity in appearance between the boundary of the turbulent fluid and the surface of a tumultuous sea is more than merely superficial.

5. Jets

In considering the two-dimensional wake we passed rather lightly over the fact that our similarity surfaces were not planes $x_2 = \text{constant}$ but were parabolic. This was justified at sufficiently large values of x_1, since these surfaces there approach closely to such planes. For jets, however, such an assumption becomes very questionable. It is found (Corrsin & Kistler 1954) that in jets characteristic scales vary as x_1, and characteristic velocities vary as x_1^{-1} for circular jets and as $x_1^{-1/2}$ for two-dimensional jets. In these flows it is obviously not possible to choose suitably a unique direction for x_2 in cartesian coordinates in order to apply Phillips's theory. The appropriate coordinate system will be spherical for the circular jet and cylindrical for the two-dimensional jet.

For the circular jet we have boundary conditions defined on a cone, with vertex at a virtual origin close to the source of the jet and angle such that it just contains all the turbulent fluid (about 25° included angle according to experimental results). In spherical coordinates (r, α, β) this cone is defined by $\alpha = \alpha_0$, and following Phillips's general line of argument we define a potential $\phi(r, \alpha, \beta)$ such that $u_\alpha = (1/r)(\partial\phi/\partial\alpha)$ is a statistically prescribed function of r and β on $\alpha = \alpha_0$. $(u_\alpha)_{\alpha=\alpha_0}$ will be characterized by a typical scale which varies as r and a typical magnitude which varies as r^{-1}. Such a function can be accurately and conveniently described by

$$(u_\alpha)_{\alpha=\alpha_0} = \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} ik_r A(k_r, k_\beta) r^{ik_r-1} \exp(ik_\beta\beta)\, dk_r, \tag{5.1}$$

where k_r and k_β are dimensionless wave-numbers, the summation is over k_β which takes only integral values, and A is complex. The potential function everywhere outside the cone will then be

$$\phi = \sum \int A(k_r, k_\beta) r^{ik_r} \exp(ik_\beta\beta) \psi(\alpha, k_r, k_\beta)\, dk_r \tag{5.2}$$

where ψ will be expressible in terms of Associated Legendre Functions of integral order k_β, pure imaginary degree ik_r, and real argument $\cos\alpha$. The function ψ may be analytically different for the various combinations of positive and negative values of k_r and k_β and will depend to some extent on other boundary conditions such as those that are imposed, for example, if the jet issues from a hole in an infinite wall.

Unfortunately there appears to be no published study of such functions, and it hardly seems worth conducting a detailed study for the present application.

6. Mean flow outside a circular jet

A similar argument to that used to derive (2.1) shows that the mean flow component U_α is given by

$$U_\alpha = Cr^{-1}, \tag{6.1}$$

where C is a negative constant on the boundary of the turbulent region $\alpha = \alpha_0$.

A solution of Laplace's equation for the mean velocity potential Φ, with (6.1) as a boundary condition and symmetry in β, is

$$\Phi = A_1 \log(r \sin\alpha) + A_2 \log(\tan\tfrac{1}{2}\alpha), \qquad (6.2)$$
$$A_1 \cos\alpha_0 + A_2 = C \sin\alpha_0.$$

Expression (6.2) leaves one arbitrary constant to be determined by other boundary conditions.

If the jet issues from a hole in an infinite wall at $\alpha = \frac{1}{2}\pi$, then $A_2 = 0$ and

$$\Phi = C \tan\alpha_0 \log(r \sin\alpha), \qquad (6.3)$$

which corresponds to a very simple uniform radial flow directed everywhere towards the axis of the jet, with speed inversely proportional to the distance from this axis.

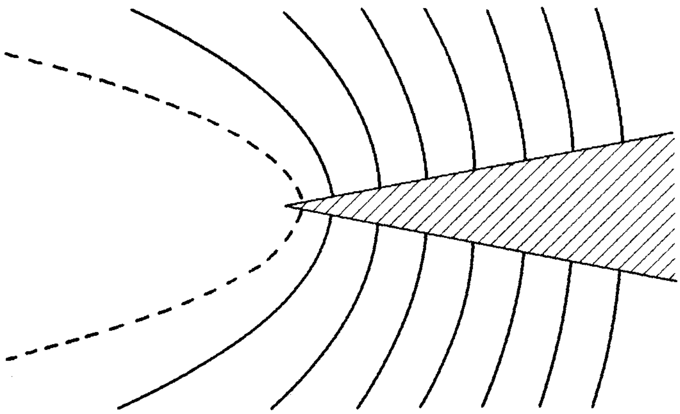


Figure 4. Flow lines into a self-preserving circular jet issuing from a nozzle.

If, as in Corrsin & Kistler's (1954) experiments, the jet issues from a nozzle, the boundary condition may be approximated by

$$\frac{\partial\Phi}{\partial\alpha} = 0 \quad \text{at} \quad \alpha = \pi$$

and we have

$$\Phi = \frac{C \sin\alpha_0}{\cos\alpha_0 + 1} \log(r \sin\alpha \tan\tfrac{1}{2}\alpha). \qquad (6.4)$$

In this case the flow lines are given by

$$r(\cos\alpha + 1) = \text{constant} \qquad (6.5)$$

and are sketched in figure 4. It will be noted that, even here, when close to the jet, the mean flow is directed essentially perpendicular to the axis of the jet.

7. Two-dimensional jet—mean flow

By a 'two-dimensional jet' we refer to the type of flow produced when fluid issues from a long uniform slot. The mean flow is two-dimensional, but fluctuating motions occur in all three dimensions. In this type of flow the similarity conditions require that the scales vary as r and the characteristic velocities as $r^{-1/2}$. In cylindrical coordinates (r, α, z) this requires that

$$U_\alpha = Dr^{-1/2} \tag{7.1}$$

on the edge of the turbulent region, $\alpha = \alpha_0$. *D is a negative constant if* α_0 is positive.

The velocity potential satisfying (7.1) and also the condition $U_\alpha = 0$ at $\alpha = \frac{1}{2}\pi$ (corresponding to a slot in an infinite wall) is

$$\Phi = Dr^{1/2}\{\cos(\tfrac{1}{2}\alpha_0 + \tfrac{1}{4}\pi)\}^{-1}\sin(\tfrac{1}{2}\alpha + \tfrac{1}{4}\pi). \tag{7.2}$$

The corresponding flow lines, which are given by

$$r^{1/2}\cos(\tfrac{1}{2}\alpha + \tfrac{1}{4}\pi) = \text{constant}, \tag{7.3}$$

are sketched in figure 5.

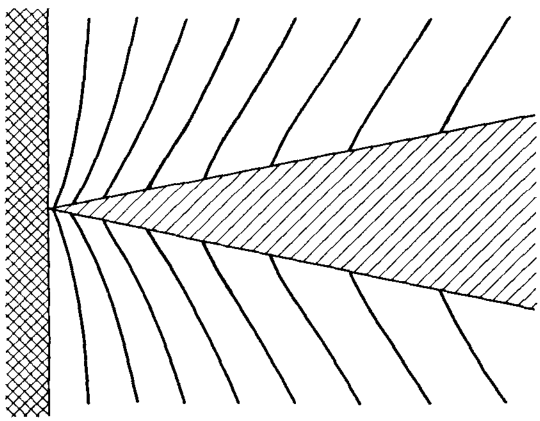


Figure 5. Flow lines into a self-preserving two-dimensional jet issuing from an infinite wall.

8. Fluctuating potential flow outside a two-dimensional jet

I have been unable to find, in three dimensions, a suitable function to describe the boundary condition for flow outside a two-dimensional jet. However, if we can imagine a flow in which the fluctuating motions are also two-dimensional the problem becomes rather simple, and the conclusions derived from a study of this hypothetical flow may, with some caution, be applied to the more complex real flows.

A velocity potential in polar coordinates (r, α), for which $\overline{u_i^2}$ varies as r^{-1} and typical scales vary as r, and which satisfies the boundary condition $u_\alpha = 0$, at $\alpha = \frac{1}{2}\pi$ (corresponding to a slot in an infinite wall), can be constructed thus:

$$\phi = r^{1/2}\int_0^\infty A\{e^{-k\alpha}\cos(B + k\log r + \tfrac{1}{2}\alpha) + e^{-k(\pi-\alpha)}\cos(B + k\log r + \tfrac{1}{2}\pi - \tfrac{1}{2}\alpha)\}\,dk, \tag{8.1}$$

where A and B are real functions of k.

Now in the region of greatest interest, $\pi - \alpha \doteqdot 3$, which is of the order of ten times as great as α. Therefore, for k greater than about one, the second term in this integral is negligible compared with the first. For small k, however, the two terms approach equality in magnitude.

For $k \ll 1$, then, the velocity components reduce to

$$\left.\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} \doteqdot r^{-1/2}\sin(\tfrac{1}{2}\alpha + \tfrac{1}{4}\pi)\int_0^\infty A\cos(B + k\log r + \tfrac{1}{4}\pi)\,dk, \\ u_\alpha &= \frac{1}{r}\frac{\partial \phi}{\partial r} \doteqdot r^{-1/2}\cos(\tfrac{1}{2}\alpha + \tfrac{1}{4}\pi)\int_0^\infty A\cos(B + k\log r + \tfrac{1}{4}\pi)\,dk. \end{aligned}\right\} \tag{8.2}$$

u_r and u_α are therefore fully correlated, and the only motions at these very small wave-numbers are along streamlines given by

$$r^{1/2}\cos(\tfrac{1}{2}\alpha + \tfrac{1}{4}\pi) = \text{constant}. \tag{8.3}$$

It will be observed that (8.3) is identical with (7.3), the expression for the streamlines of the mean flow into a two-dimensional jet. We have therefore come to the interesting, if not particularly surprising, conclusion that the very small wave-number fluctuations follow the same streamlines as the mean flow, shown in figure 5. This result is expected to be valid much more generally than just in this two-dimensional case, for temporal variations in the rate of flow from the jet will be described in just this way. It should be noted that although these fluctuations are characterized by very small wave-numbers, their frequency (in time) may be quite high.

For $k > 1$, the second term in (8.1) becomes negligible, and we have

$$\left.\begin{aligned} u_r &\doteqdot r^{-1/2}\int_0^\infty Ae^{-k\alpha}\{\tfrac{1}{2}\cos(B + k\log r + \tfrac{1}{2}\alpha) - \\ &\qquad\qquad - k\sin(B + \log r + \tfrac{1}{2}\alpha)\}\,dk, \\ u_\alpha &\doteqdot -r^{-1/2}\int_0^\infty Ae^{-k\alpha}\{\tfrac{1}{2}\sin(B + k\log r + \tfrac{1}{2}\alpha) + \\ &\qquad\qquad + k\cos(B + k\log r + \tfrac{1}{2}\alpha)\}\,dk. \end{aligned}\right\} \tag{8.4}$$

$B(k)$ is a rapidly changing function of k satisfying the inequality

$$-\pi < B < \pi,$$

and if we average over different flow realizations we would expect that

$$\overline{\sin\{B(k_1) \pm B(k_2)\}} = \overline{\cos\{B(k_1) \pm B(k_2)\}} = 0,$$

unless $k_1 = k_2$. B will take all values equally probably in different flow realizations, so

$$\left.\begin{aligned} \overline{u_r^2} = \overline{u_\alpha^2} &= \tfrac{1}{2}r^{-1}\int_0^\infty A^2(\tfrac{1}{4} + k^2)e^{-2k\alpha}\,dk, \\ \overline{u_r u_\alpha} &= 0. \end{aligned}\right\} \tag{8.5}$$

Thus $\overline{u_r^2} = \overline{u_\alpha^2}$, and u_r and u_α are uncorrelated, and (8.5) is obviously the two-dimensional analogue in polar coordinates of Phillips's results (1.1) and (1.2).

The scale of a motion characterized by $k = 1$ will be such that $\log r$ changes by about $\frac{1}{2}\pi$, that is r changes by a factor of the order of 5. Thus

$k = 1$ is a very small wave-number indeed, and virtually all the energy of the velocity fluctuations may be in the range of wave-numbers for which (8.5) is applicable.

Although a detailed analysis of the actual case of three-dimensional fluctuations in the two-dimensional jet has not been found possible, some of its probable characteristics can be shown.

It can readily be demonstrated that for irrotational flow in which mean quantities are invariant in the z-direction the cylindrical coordinate equivalents to equations (3.8) are

$$\left.\begin{aligned} \frac{\partial}{\partial r}(\tfrac{1}{2}q^2-\overline{u_r^2}) &= \frac{1}{r}\frac{\partial}{\partial \alpha}\overline{u_r u_\alpha}+\frac{\overline{u_r^2}-\overline{u_\alpha^2}}{r}, \\ \frac{\partial}{\partial \alpha}(\tfrac{1}{2}q^2-\overline{u_\alpha^2}) &= 2\overline{u_r u_\alpha}+r\frac{\partial}{\partial r}\overline{u_r u_\alpha}. \end{aligned}\right\} \tag{8.6}$$

Our similarity conditions permit us to write

$$q^2 = r^{-1}p^2, \qquad u_i^2 = r^{-1}v_i^2, \tag{8.7}$$

where p and $\overline{v_i v_j}$ are independent of r. Equations (8.6) then become

$$\left.\begin{aligned} \overline{v_\alpha^2}-\tfrac{1}{2}p^2 &= \frac{\partial \overline{v_r v_\alpha}}{\partial \alpha}, \\ \overline{v_r v_\alpha} &= \frac{\partial}{\partial \alpha}(\tfrac{1}{2}p^2-\overline{v_\alpha^2}). \end{aligned}\right\} \tag{8.8}$$

In view of (1.8) and (8.5), the most likely solution to these equations is

$$\overline{v_\alpha^2} = \tfrac{1}{2}p^2, \qquad \overline{v_r v_\alpha} = 0. \tag{8.9}$$

It would seem probable then that there is no Reynolds shear stress across surfaces α = constant in the two-dimensional jet. This is a result of the behaviour $q^2 \propto r^{-1}$ and would not be possible with any other dependence on r, unless $u_z \equiv 0$. It is in contrast to the situation in the wake, where we saw that a shear stress does occur.

9. Concluding note

This study, as far as it has gone, has shown that the results of Phillips's simple model of irrotational flow outside a turbulent region are not greatly affected by introducing simple inhomogeneities into the boundary conditions. It has been demonstrated that these irrotational fluctuations are capable in some circumstances of producing a Reynolds shear stress, but the basic conclusion that the velocity component directed away from the boundary of the turbulence contains about half the total energy in the fluctuations seems to be firmly established. Still unresolved then, is the contradiction pointed out by Townsend (1956, p. 191) between this kind of theory and the experimental results which indicate that in jets the downstream component is larger than this cross-stream component. Although it is not altogether satisfactory to rely on the extensibility of the results of the two-dimensional analysis of § 8 to real three-dimensional cases, flow similar to that described by (8.2) is intuitively likely and (8.5) is very similar to the three-dimensional result (1.1). It seems reasonable to infer that in three-dimensional cases there will also be two basic modes to the irrotational fluctuations: very low wave-number motions along the mean flow lines, and higher wave-number motions similar in structure to those in gravity waves. Figures 4 and 5 and equations (1.1) and (8.9) show that both modes give greater contributions to the cross-stream than to the down-stream motions.

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